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Stochastic Differential Equations in Quantum Statistical Mechanics. Observables and Multiple Wiener Integrals

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The physical and mathematical framework for quantum mechanical stochastic differential equations is discussed as the quantization of c-number equations that typically describe Brownian motion in polynomial potentials.

KEY WORDS: Quantum Brownian motion; stochastic differential equation; quantum noise; fluctuation-dissipation relation; heat bath; Wick polynomial; multiple Wiener integral.

1. MOTIVATION

Let S be a quantum mechanical system in thermal equilibrium, and A be one (or a few) of its observables. When other degrees of freedom in S are left unspecified, the time evolution (or its suitable modification) $A(t)$ of A forms what we might generally define as a stochastic process in quantum statistical mechanics. To obtain useful insights, however, this general definition needs to be restricted to a narrower range of $\{S, A\}$'s that give simple, but hopefully rich structures. In the corresponding classical problems such rich structures are presented by diffusion processes generated from Gaussian white noise through stochastic differential equations (SDEs). This work discusses aspects of the quantization of the further subclass composed of Brownian motion processes in polynomial potentials.

The research into modeling Brownian motion processes by classical mechanical systems was initiated decades ago.[2] Hamiltonian systems are

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[2] Pioneering works related to this research are cited in the bibliographies of Refs. 1–4.

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known[1-4] with N degrees of freedom composed of harmonic oscillators in thermal equilibrium and coupled to a particle in an extra (possibly anharmonic) force field: As $N \to \infty$, this particle realizes the mentioned type of Brownian motion if harmonic oscillators tend to form, roughly speaking, an elastic string. This particular limit $N \to \infty$ will be called the continuum limit; Sections 2 and 3 will give its precise characterization. Ford *et al.*[3] advanced a decisive step by quantizing a model system with the particle in its own harmonic potential, performing the continuum limit on the solution for the particle and deriving the quantized Ornstein–Uhlenbeck (Q-OU) process. More recently, the quantization research was resumed[5,6] by taking the continuum limit Q-OU problem as unperturbed and adding perturbations in the generator of the quantum dynamics: The analyses culminated in a proof[6,16] that the perturbed dynamics can be described by the same form of quantized SDEs (Q-SDEs) as the corresponding classical SDEs, for anharmonic perturbation potentials that satisfy a few boundedness and differentiability conditions.

However, the continuum limit Q-OU problem and its generalizations involve a few conceptual difficulties that originate in the quantized Gaussian white noise[3,5 7] (Q-noise, for short). A Q-noise $w(t)$ exists as a quasi-free field,[11,12] and makes sense of an observable only after smearing. This $w(t)$ the takes position of a random force in a Q-SDE, and its presence also deprives the momentum of the Brownian particle of the sense of an observable at respective t. This circumstance raises some fundamental questions that still seem to lack explicit answers: Is this *particle* a real physical existence, or more pragmatically, what type of quantities should a Q-SDE evaluate?

A way to deal with these points is to go back to finite systems and consider the implications of the continuum limit on them in more detail.[3] These form precisely the aim of this work. We reconsider structures of the above-mentioned models, and also look for a representation of the continuum limit on these models. The paper obtains the following results.

1. A class of nonlinear, Langevin-type equations are shown to exist as precontinuum limits, giving reduced Heisenberg equations[4] that retain the same forms as classical equations of motion. This will be seen to suggest the existence of quantum heat reservoirs, with their structures not dependent on systems attached to them, as pairs of a quantum Gaussian operator process and a dissipation term that are linked together by a fluc-

[3] In this regard, the convergence of the quantum dynamics itself in the continuum limit was discussed recently for bounded but not necessarily differentiable potentials.[17]

[4] Heisenberg equations for relevant dynamical variables only, other degrees of freedom being contracted into a random force and a dissipation term; cf. Sections 2 and 3.

tuation-dissipation relation, and with the former of satisfying the KMS property with respect to the time shift.

2. The Q-noise, which is singular but well-defined,[3,5-7] is shown to give a representation of a class of quantum Gaussian processes that include noise forces in result 1. This representation is also shown to realize the continuum limit on the heat reservoir in a unified and tractable way.

3. A few special structures are shown to exist in Wick polynomial *moving averages* of the Q-noise, assuring the convergence of some class of covariance functions generated from the quantum noise of result 1 in the continuum limit.

These results will clarify the basic mode of existence of a quantized SDE in its relation to the classical one and a range of observable quantities to be evaluated by a Q-SDE, together with some further perspectives on remaining problems and possible extensions.

Section 2 exposes the classical model and the continuum limit on it. The quantization is discussed in Section 3. Structures in the Q-noise and the associated Wick polynomials are given in Sections 4 and 5. The general framework of a nonlinear Q-SDE will be concluded in Section 6 on the basis of these. A few details in the analysis are deferred to Appendices A–C.

2. CLASSICAL MODELS AND THE REDUCED EQUATION OF MOTION

Let $\{Q, P\}$ be the coordinate and the momentum of a particle of mass m in a differentiable potential $V(Q)$ and interacting with K harmonic oscillators with angular frequencies $\{\omega_k > 0; 1 \leqslant k \leqslant K\}$. We consider first the Hamiltonian

$$H = P^2/2m + V(Q) + \frac{1}{2} \sum_{k=1}^{K} [p_k^2/m_k + m_k \omega_k^2 (q_k - Q)^2 - \hbar\omega_k] \quad (2.1)$$

and the associated classical canonical equations

$$dq_k/dt = p_k/m_k, \qquad dp_k/dt = -m_k \omega_k^2 [q_k - Q(t)] \quad (2.2)$$

$$dQ/dt = P/m, \qquad dP/dt = -V'(Q) + \sum_{k=1}^{K} m_k \omega_k^2 [q_k(t) - Q(t)] \quad (2.3)$$

Let $T < 0$ denote the initial time; later we let $T \to -\infty$. The solution for $q_k(t)$ is

$$q_k(t) = q_k(T) \cos[\omega_k(t-T)] + (m_k\omega_k)^{-1} p_k(T) \sin[\omega_k(t-T)]$$
$$+ \omega_k \int_T^t \sin[\omega_k(t-s)]\, Q(s)\, ds$$

Partial integration gives

$$m_k \omega_k^2 [q_k(t) - Q(t)] = \sigma f_k(t) - \int_T^t \gamma_k(t-s)\, P(s)\, ds$$

where $\sigma > 0$ is a constant to be fixed later and

$$\begin{aligned} \gamma_k(\tau) &= (m_k \omega_k^2/m) \cos(\omega_k \tau) \\ f_k(t) &= (m_k \omega_k^2/\sigma)[q_k(T) - Q(T)] \cos[\omega_k(t-T)] \\ &\quad + (\omega_k/\sigma)\, p_k(T) \sin[\omega_k(t-T)] \end{aligned} \tag{2.4}$$

These are substituted back into (2.3). Then we have

$$dQ/dt = P/m,$$

$$dP/dt = -V'[Q(t)] - \int_T^t \tilde{\gamma}(t-s)\, P(s)\, ds + \sigma \tilde{w}(t) \tag{2.5}$$

$$\tilde{\gamma}(\tau) = \sum_{k=1}^{K} \gamma_k(\tau) = \int_0^\infty \cos(\omega\tau)\, \rho(\omega)\, d\omega,$$

$$\rho(\omega) = \sum_{k=1}^{K} (m_k \omega_k^2/m)\, \delta(\omega - \omega_k) \tag{2.6}$$

$$\tilde{w}(t) = \sum_{k=1}^{K} f_k(t) \tag{2.7}$$

In general, $\tilde{\gamma}(\tau)$ is almost periodic with a pure point spectral density $\rho(\omega)$.

No statistics was involved in this derivation of (2.5). Now we pose the initial probability density $\propto e^{-\beta H}$ for an inverse temperature $\beta > 0$, denoting an expectation as $\langle \cdots \rangle_c$. This density has factors depending on elements of the initial data $D = \{Q(T),\ P(T),\ q_k(T) - Q(T),\ p_k(T);\ 1 \leqslant k \leqslant K\}$, respectively, all with Gaussian forms, except the one on $Q(T)$. Thus, the $f_k(t)$ are Gaussian and mutually independent, with

$$\langle f_k(t) \rangle_c = 0, \qquad \langle f_k(t)\, f_l(t+\tau) \rangle_c = \delta_{kl} \gamma_k(\tau)/\beta\sigma^2 \tag{2.8}$$

The stochastic process $\tilde{w}(t)$ is independent of $\{Q(T), P(T)\}$, stationary, and Gaussian, satisfying a fluctuation-dissipation relation,[5]

[5] There are ways[3,4,8] to reduce equations of motion (2.2)–(2.3). The present (2.5)–(2.9) are after Ref. 4. The general theory of Mori[8] gives (A. Sakurai, private communication), with a particular choice of the projection in the formalism, the same result on (2.1)–(2.3) if $V(Q)$ is harmonic, but departs from (2.5) otherwise.[4]

$$\langle \tilde{w}(t) \rangle_c = 0$$

$$\langle \tilde{w}(t)\, \tilde{w}(t+\tau) \rangle_c \equiv C(\tau) = m\tilde{\gamma}(\tau)/\beta\sigma^2 \equiv 2 \int_0^\infty \cos \omega\tau \, d\Gamma(\omega) \equiv C(-\tau) \qquad (2.9)$$

$$\Gamma(\omega) \equiv \sum_{\omega_k \leqslant \omega} m_k \omega_k^2 / 2\beta\sigma^2 = \int_0^\omega m\rho(\omega')\, d\omega'/2\beta\sigma^2, \qquad \Gamma(0) = 0$$

The continuum limit may now be defined. Take a sequence for $\Lambda = 1, 2,...$ of sets $\{m_k^{(\Lambda)} > 0, \omega_k^{(\Lambda)} > 0; 1 \leqslant k \leqslant K(\Lambda) < \infty\}$. Define a Hamiltonian $H^{(\Lambda)}$ and a step function $\Gamma^{(\Lambda)}(\omega)$ by replacing $\{m_k, \omega_k\}$ with $\{m_k^{(\Lambda)}, \omega_k^{(\Lambda)}\}$ in (2.1) and in $\Gamma(\omega)$ of (2.9), respectively. Assume that $\Gamma^{(\Lambda)}(\omega) \uparrow m\gamma\omega/\pi\beta\sigma^2$ holds as $\Lambda \to \infty$ at $\forall\omega \in \mathbb{R}_+$, where $\uparrow$ is the convergence from below and a constant $\gamma > 0$ is defined by this relation. The sequence $\{H^{(\Lambda)}\}$ of Hamiltonian systems are then named[6] to tend to a continuum. Each $H^{(\Lambda)}$ gives Λ-dependent structures, typically expectation functions or functionals (EFs) formed with $\{Q(t), P(t), \tilde{\gamma}(\tau), \tilde{w}(t)\}$. Limits for $\Lambda \to \infty$ of these EFs [and also of commutation relations (CRs) in quantum mechanics] form the contents of the continuum limit. If mechanical systems or stochastic processes realize these limits of EFs, such systems or processes will be called the continuum limit of $\{H^{(\Lambda)}\}$.

Roughly speaking, this definition implies

$$\rho^{(\Lambda)}(\omega) = \sum_{k=1}^{K(\Lambda)} m_k^{(\Lambda)} (\omega_k^{(\Lambda)})^2 \, \delta(\omega - \omega_k^{(\Lambda)}) \to \text{const} \qquad \text{as} \quad \Lambda \to \infty$$

and assures by (2.9) that $\tilde{w}(t)$ converges to a Gaussian white noise $w(t)$ with $\langle w(t) \rangle_c = 0$ and

$$\langle w(t)\, w(t+\tau) \rangle_c = 2m\gamma\delta(\tau)/\beta\sigma^2$$

When $V(Q)$ is harmonic, $\{Q(t), P(t)\}$ was shown to converge to an OU process in this limit.[1-4] By Λ-uniform bounds on the $\langle |Q^m P^n| \rangle_c$ given by the probability density $\propto \exp(-\beta H^{(\Lambda)})$, we further have (see footnote 6): Expectation values of polynomials of $\{Q(t), P(t)\}$ of (2.5) for an arbitrary polynomial $V(Q) \geqslant 0$ converge in the continuum limit, for $T \leqslant t \leqslant \forall T' < \infty$,

[6] Some mathematical restrictions must further be posed. Double limits, in the first of which $\Gamma^{(\Lambda)}(\omega)$ tends to an absolutely continuous $\Gamma_\Omega(\omega)$ with $\Gamma'_\Omega(\omega) \equiv 0$ for $\omega > \exists\Omega$, and in the second $\Omega \to \infty$ [cf. example below (3.6)], realize them feasibly in a physical way. The first limit is ruled by Lemma 3 of Ref. 17 and the second may be treated by (a classical version of) Lemma 1 in Section 4. A single sequence fulfilling the restrictions may then be extracted from these two as in the text.

to those given by the solution of the SDE for the above Gaussian white noise $w(t)$,

$$dQ/dt = P/m, \qquad dP/dt = -V'[Q(t)] - \gamma P(t) + \sigma w(t) \tag{2.10}$$

Hereafter we drop the superscript (Λ), and take a time scale such that $\beta\sigma^2 = 2m\gamma$ in (2.10). By this choice, $\tilde{w}(t)$ becomes a standard Gaussian white noise $w(t)$ in the continuum limit $\Gamma(\omega) \uparrow \omega/2\pi$, with

$$\langle w(t) \rangle_c = 0, \qquad \langle w(t)\, w(t+\tau) \rangle_c = \delta(\tau) \tag{2.11}$$

in the nonlinear SDE (2.10). Since the frictional force is $-\kappa\, dQ/dt = -\gamma P$, Einstein's relation is independent of m in terms of the friction constant $\kappa = m\gamma$.

3. QUANTUM MECHANICAL MODELS

The Hamiltonian H in (2.1) for $K < \infty$ is now interpreted as a quantum mechanical operator, with the Schrödinger representation on $C_0^\infty(\mathbb{R}^{K+1})$ in mind. The stability condition $V(Q) \geqslant 0$ is assumed hereafter on the potential. Then H is essentially self-adjoint[9] or defines a unique quantum dynamics $\exp[-i(t-T)H/\hbar]$, and the Heisenberg picture gives equations of motion for $\{q_k, p_k; 1 \leqslant k \leqslant K\}$ that are identical with the classical (2.2), because H is quadratic in $\{q_k - Q, p_k; 1 \leqslant k \leqslant K\}$. Therefore, the reduced equation (2.5) for $\{Q(t), P(t)\}$ remains valid in this picture irrespective of the form of $V(Q)$, with $\tilde{w}(t)$ and $\tilde{\gamma}(\tau)$ given by the same (2.6), (2.7), and (2.4).

There arises, however, one quantal complication. If $V(Q)$ is nonharmonic, the density matrix $\propto e^{-\beta H}$ posed on elements of $D = \{Q(T), P(T), q_k(T) - Q(T), p_k(T); 1 \leqslant k \leqslant K\}$ does not assure independence and Gaussian character of $\{q_k(T) - Q(T), p_k(T); 1 \leqslant k \leqslant K\}$ by the noncommutativity of $q_k(T) - Q(T)$ with $P(T)$ and $p_k(T)$. Thus, $\tilde{w}(t)$ cannot be inferred to have Gaussian expectations independently of $V(Q)$. In order to circumvent this complication, we adopt a modified model. The harmonic Hamiltonian with a certain $\omega_0 > 0$,

$$H_0 = P^2/2m + m\omega_0^2 Q^2/2 + \frac{1}{2} \sum_{k=1}^{K} [p_k^2/m_k + m_k \omega_k^2 (q_k - Q)^2 - \hbar\omega_k] \tag{3.1}$$

is assumed to govern the dynamics for $t < T$, and elements of D are posed of the statistics of the density matrix $\propto \exp(-\beta H_0)$. The relevant quantal expectations will be denoted by $\langle \cdots \rangle$. For $t \geqslant T$ the dynamics is switched to $\exp[-i(t-T)H/\hbar]$ as before. These devices assure (2.4), (2.6), and (2.7) again for the reduced equation of motion or the quantum stochastic integrodifferential equation (Q-SIDE) (2.5) for $t \geqslant T$, and assign jointly (ordered) Gaussian expectations on $\{\tilde{w}(t), Q(T), P(T); t \geqslant T\}$.

A point to be stressed is that this model gives Q-SIDE (2.5) with the same memory kernel $\tilde{\gamma}(\tau) = 2\gamma C(\tau)$, where $C(\tau)$ is the *classical* equilibrium covariance function of the random force,

$$\langle \tilde{w}(t)\, \tilde{w}(t+\tau) \rangle_c = C(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\, d\Gamma(\omega), \qquad \Gamma(-\omega) \equiv -\Gamma(\omega) \quad (3.2)$$

The classical expectation $\langle \cdots \rangle_c$ may be taken with any of the two probability densities $\propto \exp(-\beta H)$ or $\exp(-\beta H_0)$. This structure in Q-SIDE (2.5) will again be called the fluctuation-dissipation relation. We define, as usual, $[A, B] \equiv AB - BA$ and $A \circ B \equiv (AB + BA)/2$. Besides $[Q(T), P(T)] = i\hbar$, there hold, by (2.4), (2.6), (2.7), and (2.9),

$$[\tilde{w}(s), \tilde{w}(t)] = i\beta\hbar C'(s-t) = -\int_{-\infty}^{\infty} e^{i\omega(s-t)} \beta\hbar\omega\, d\Gamma(\omega) \qquad (3.3)$$

$$\langle \tilde{w}(t) \rangle = 0, \qquad \langle \tilde{w}(s) \circ \tilde{w}(t) \rangle = C_{\beta\hbar}(s-t) \qquad (3.4)$$

$$C_{\beta\hbar}(\tau) \equiv \int_{-\infty}^{\infty} e^{i\omega\tau} \mu(\omega)\, d\Gamma(\omega), \qquad \mu(\omega) \equiv \tfrac{1}{2}\beta\hbar\omega \coth(\tfrac{1}{2}\beta\hbar\omega) \qquad (3.5)$$

$$[\tilde{w}(t), Q(T)] = 0, \qquad [\tilde{w}(t), P(T)] = -i\hbar C(t-T) \qquad (3.6)$$

together with the ordered Gaussian law for higher order moments of $\tilde{w}(t)$.

In order to prepare for the the quantal continuum limit, we now consider a limit $K \to \infty$ in which a pure point-type $\Gamma(\omega)$ converges to an absolutely continuous but bounded form with $\int_{-\infty}^{\infty} |\omega|\, d\Gamma(\omega) < \infty$, giving sense to (3.3)–(3.5). We take for convenience the following special Hamiltonian H with $\tilde{q}_0 \equiv Q$ and constants $\tilde{m}$, $\tilde{\omega} > 0$:

$$H = P^2/2m + V(Q) + \frac{1}{2} \sum_{k=1}^{K} [\tilde{p}_k^2/\tilde{m} + \tilde{m}\tilde{\omega}^2(\tilde{q}_k - \tilde{q}_{k-1})^2] - \hbar c_K \qquad (3.7)$$

where c_K is a suitable constant. A linear transformation of $\{\tilde{q}_k, \tilde{p}_k; 1 \leqslant k \leqslant K\}$ brings (3.7) to the form of (2.1),[7] with

$$\omega_k \equiv \omega(\theta) = \omega_L \sin(\theta/2), \qquad \theta = \pi(2k-1)/(2K+1)$$
$$1 \leqslant k \leqslant K, \quad \omega_L = 2\tilde{\omega} \qquad (3.8)$$

[7] Denote the sum in (3.7) as ${}^t\tilde{\mathbb{p}}\tilde{\mathbb{p}}/\tilde{m} + {}^t\tilde{\mathbb{q}}' U \tilde{\mathbb{q}}'$, with t denoting transpose, ${}^t\tilde{\mathbb{p}} = (\tilde{p}_1, \ldots, \tilde{p}_K)$ and ${}^t\tilde{\mathbb{q}}' = {}^t\tilde{\mathbb{q}} - {}^t\mathbb{1}Q = (\tilde{q}_1 - Q, \ldots, \tilde{q}_K - Q)$. Let $S = (S_{kl})$ be an orthogonal matrix that gives a diagonal $\Omega = {}^tSUS$, and define ${}^t\bar{\mathbb{q}} = (\bar{q}_1, \ldots, \bar{q}_K) = {}^t\tilde{\mathbb{q}}S$, ${}^t\bar{\mathbb{p}} = {}^t\tilde{\mathbb{p}}S$. Diagonal elements of Ω are seen to be given by $\{\omega_k\}$ of (3.8), and furhter transformations $q_k = \bar{q}_k/N_k$, $p_k = \bar{p}_k N_k$, $N_k = \sum_{l=1}^{K} S_{lk}$ yield (2.1).

Therefore, Q-SIDE (2.5) also holds for $\{Q(t), P(t)\}$. A merit of (3.7) is that $\lim_{K\to\infty} \tilde{\gamma}(\tau) = (\tilde{m}\omega_L/m)\, J_1(\omega_L \tau)/\tau$ is known [cf. Ref. 4, Eqs. (1), (13), (24), and (26)] with Bessel function J_1, a result due also to Sakurai (cf. footnote 5). This gives

$$\lim_{K\to\infty} d\Gamma(\omega)/d\omega = \begin{cases} (\tilde{m}\omega_L/2\pi m\gamma)[1-(\omega/\omega_L)^2]^{1/2}, & |\omega| \leqslant \omega_L \\ 0, & \text{otherwise} \end{cases} \quad (3.9)$$

The transition $K\to\infty$ on $\Gamma(\omega)$ thus realizes the irreversibility $\tilde{\gamma}(\tau)\to 0$ as $|\tau|\to\infty$.[8] The continuum limit on (3.7) is also seen to be realized by a further limit $\omega_L = 2\tilde{\omega}\to\infty$ with the mass ratio $m/\tilde{m} = \omega_L/\gamma\to\infty$.

The quantum dynamics for $K=\infty$ of the lattice system (3.7) may be defined as follows. Let $\mathscr{A} \equiv \{\xi = (\xi_0, \xi_1,...);\ \xi_k \in \mathbb{R}\}$ be the set of finite sequences with $\xi_k \neq 0$ only for $k < \exists K' < \infty$. Define

$$\begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} (\xi, t) \equiv \sum_k \xi_k \begin{pmatrix} \tilde{q}_k \\ \tilde{p}_k \end{pmatrix} (t)$$

with $\xi \in \mathscr{A}$ as a test function. We put $V(Q)\to V_0(Q) = m\omega_0^2 \tilde{q}_0^2/2$ for a while, and define the would-be EFs and CRs of the infinite harmonic lattice $\{(^{\tilde{q}}_{\tilde{p}})(\xi, t);\ \xi\in\mathscr{A},\ t<T\}$ as limits for $K\to\infty$ of those given by (3.7) for this harmonic case in equilibrium. These limits are suggested to exist (as an implication of Ref. 3), and we simply assume their existence. Limit EFs and CRs should inherit Gaussian (harmonic or free) and various positivity properties, together with the KMS property with respect to the time evolution described by the infinite system of linear Heisenberg equations as the limit $K\to\infty$ of those given by (3.7). The possibility of the reconstruction of an infinite harmonic lattice on these EFs and CRs is of no doubt. In particular, the modular operator $\Delta > 0$ should exist, implementing unitarily the mentioned harmonic time evolution as, e.g., $\tilde{q}_k(t) = \Delta^{i(t-T)} \tilde{q}_k(T)\, \Delta^{-i(t-T)}$, where $\tilde{q}_k(t) = \tilde{q}(\varepsilon^{(k)}, t)$ for $\varepsilon^{(k)} = (\varepsilon_0^{(k)}, \varepsilon_1^{(k)},...)$, with $\varepsilon_l^{(k)} = \delta_{kl}$. Define $G = -\hbar \log \Delta$, and from $t = T$ on consider the time evolution generated from *any* self-adjoint extension of $H_\infty = G + V[\tilde{q}_0(T)] - V_0[\tilde{q}_0(T)]$ as the Hamiltonian. Assuming the absence of domain problems, we obtain Heisenberg equations for $\{(^{\tilde{q}_k}_{\tilde{p}_k})(t);\ t>T\}$ that correspond to the formal Hamiltonian (3.7) with $K=\infty$ and a polynomial $V(Q)$. They should in turn give the reduced Q-SIDE (2.5) for (3.9).

Though the above argument is only intuitive and leaves some crucial points unproved, the interrelation of Q-SIDE (2.5) and (3.7) for $K=\infty$ might safely be taken to have been established for $\Gamma(\omega)$ of (3.9). Hereafter

[8] By the Riemann–Lebesgue lemma; (3.9) gives $C(\tau) = O(|\tau|^{-3/2})$ for $|\tau|\to\infty$.

we take (2.5) for an arbitrary, absolutely continuous $\Gamma(\omega)$ with $\int_{-\infty}^{\infty} |\omega|\, d\Gamma(\omega) < \infty$ as our starting point, and discuss the convergence of a certain range of quantities formed with $\tilde{w}(t)$ in the remaining limit $\Gamma(\omega) \uparrow \omega/2\pi$, expecting perturbational construction of solutions of (2.5). Since the corresponding $C(\tau)$ vanishes as $|\tau| \to \infty$ for this type of $\Gamma(\omega)$, $T \to -\infty$ may furhter be taken on (2.5) to obtain

$$dQ/dt = P/m, \qquad dP/dt = -V'(Q) - 2\gamma \int_{-\infty}^{t} C(t-s)\, P(s)\, ds + \sigma\tilde{w}(t) \quad (3.10)$$

which will describe stationary states. In (3.10) $\tilde{w}(t)$ is characterized by (3.2)–(3.5). Lewis and Thomas[10] have shown that (3.2)–(3.5) and $\int_{-\infty}^{\infty} |\omega|\, d\Gamma(\omega) < \infty$ specify [including cases of pure point $\Gamma(\omega)$] exactly a class of Gaussian operator processes that have β-KMS property with respect to the time shift τ_t: $\tilde{w}(s) \to \tilde{w}(s+t)$. The present section indicates, therefore, the existence of a class of nonlinear Q-SIDEs satisfying a fluctuation-dissipation relation (3.2) whose noise force terms fall in the Lewis–Thomas class. The KMS property of $\tilde{w}(t)$ will be inferred in the next section from a different point of view.

4. QUANTIZED GAUSSIAN WHITE NOISE AND ITS STRUCTURES

We now construct a useful representation of the quantum Gaussian operator process $\tilde{w}(t)$ of the previous section in terms of Q-noise. Heuristically, the Q-noise $w(t)$ is obtained by formal applications of the continuum limit $d\Gamma(\omega) \to d\omega/2\pi$ on (3.3)–(3.5) as the would-be limit of $\tilde{w}(t)$. Thus,

$$[w(s), w(t)] = i\beta\hbar\delta'(s-t) \quad (4.1)$$

$$\langle w(s) \rangle = 0$$
$$\langle w(s) \circ w(t) \rangle = \delta_{\beta\hbar}(s-t) \equiv \int_{-\infty}^{\infty} e^{i\omega(s-t)} \mu(\omega)\, d\omega/2\pi \quad (4.2)$$

Introduce a still formal notation,

$$w(\xi) = \int_{-\infty}^{\infty} \xi(t)\, w(t)\, dt \quad (4.3)$$

with a smearing or test function $\xi(t)$, and define Fourier transforms by[9]

$$\Xi(\omega) \equiv \mathscr{F}[\xi(t)] \equiv (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\omega t}\xi(t)\, dt \tag{4.4}$$

Then (4.1) and (4.2), supplemented with the ordered Gaussian law, take the form[10]

$$[w(\xi), w(\eta)] = 2i\sigma(\xi, \eta), \qquad \sigma(\xi, \eta) \equiv i \int_{-\infty}^{\infty} \Xi^*(\omega)\, H(\omega)\, \tfrac{1}{2}\beta\hbar\omega\, d\omega \tag{4.5}$$

$$\langle e^{iw(\xi)} \rangle = e^{-s(\xi,\xi)/2}, \qquad s(\xi, \eta) \equiv \int_{-\infty}^{\infty} \Xi^*(\omega)\, H(\omega)\, \mu(\omega)\, d\omega \tag{4.6}$$

Quantized Gaussian white noise (Q-noise) $w(t)$ exists precisely that gives (4.5) and (4.6) (see Appendix A).[(6,7,12)] Here we need to note simply that the Q-noise $w(\xi)$ is a real, linear mapping from a test function $\xi(t)$ in the class

$$\mathscr{T} = \{\xi(t); \Xi(\omega) = \Xi^*(-\omega) \in L^2[\mathbb{R}, \mu(\omega)\, d\omega]\} \tag{4.7}$$

to a class of self-adjoint operators $\{w(\xi); \xi \in \mathscr{T}\}$ with a formal notation (4.3).

A utility of the Q-noise is manifested in the following.

Lemma 1. Let $C(\tau)$ be a positive-definite function with the spectral decomposition of Bochner,

$$C(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\, d\Gamma(\omega), \quad \int_{-\infty}^{\infty} |\omega|\, d\Gamma(\omega) < \infty, \quad \Gamma(-\omega) = -\Gamma(\omega) \tag{4.8}$$

Assume that $\Gamma(\omega)$ has a density $\Gamma'(\omega) = \Gamma'(-\omega) = d\Gamma(\omega)/d\omega \geqslant 0$, and define

$$E(\omega) \equiv e^{i\chi(\omega)}[\Gamma'(\omega)]^{1/2} \equiv (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\omega\tau}\varepsilon(\tau)\, d\tau$$
$$\chi(\omega) = -\chi(-\omega): \text{ real and arbitrary} \tag{4.9}$$

Then $\varepsilon(\tau) \in \mathscr{T}$, and the moving average[11]

$$w_\varepsilon(t) \equiv w\{\mathscr{F}^{-1}[e^{-i\omega t}E^*(\omega)]\} \equiv \int_{-\infty}^{\infty} \varepsilon(t-s)\, w(s)\, ds \tag{4.10}$$

[9] Test functions on the real (t-) space are denoted hereafter by the lower case Greek letters $\xi(t)$, $\eta_k(t)$,..., and their Fourier transforms given by (4.4) are represented (often without comments) by the corresponding capital letters $\Xi(\omega)$, $H_k(\omega)$,... .

[10] The factor 2 on the rhs of $[w(\xi), w(\eta)]$ is in accordance with Manuceau.[(11)] See also Ref. 12.

[11] By the definition $w(\varepsilon) = \int_{-\infty}^{\infty} \varepsilon(s)\, w(s)\, ds$, $E^*(\omega)$ is needed in (4.10).

satisfies (3.3)–(3.5). The quantal Gaussian stationary process $w_\varepsilon(t)$ has the β-KMS property with respect to the time shift $\tau_t w_\varepsilon(s) = w_\varepsilon(s+t)$.

Proof. Integrability of $|\omega|\, \Gamma'(\omega)$ implies $E(\omega) \in L^2[\mathbb{R}, \mu(\omega)\, d\omega]$. Thus, $\varepsilon(\tau) \in \mathscr{T}$, and $w_\varepsilon(t)$ is well-defined. The commutation relation (3.3) follows from (4.5), while (3.4), (3.5), the Gaussian property, and stationarity are seen from (4.6) with the aid of (4.5). Since τ_t on $w_\varepsilon(s) = \int_{-\infty}^{\infty} \varepsilon(s-s')\, w(s')\, ds'$ acts as the time shift $w(s') \to w(s'+t)$ on the Q-noise, the β-KMS property of $w_\varepsilon(t)$ is a special case of Proposition A in Appendix A. ∎

If $C(\tau)$ of (4.8) is identified with that in (3.2), Lemma 1 asserts that $w_\varepsilon(t)$ gives a representation of the processes $\tilde{w}(t)$ of (3.3)–(3.5). Lemma 1 also enables us to represent the quantal continuum limit $\tilde{w}(t) \to w(t)$ by simply taking $w_\varepsilon(t)$ of (4.10) for $\tilde{w}(t)$ and letting $\varepsilon(\tau) \to \delta(\tau)$. The merit of this representation is that the limit is realized in one and the same Hilbert space, corresponding to the classical representation of stochastic processes in a probability space.[17] The range of operator processes represented by $w_\varepsilon(t)$ is narrower than the Lewis–Thomas class[10] in that $\Gamma(\omega)$ is restricted to have a density. The phase factor $\chi(\omega)$ may play a significant role in determining the form of $\varepsilon(\tau)$. For example, some form of $\chi(\omega)$ may give a causal $\varepsilon(\tau)$ that vanishes for $\tau < 0$, by making $E(\omega)$ analytic in the upper half ω-plane. For a Gaussian white noise $w(t)$, (4.10) gives a representation of the classical $\tilde{w}(t)$ of (2.9).

5. MOVING AVERAGES IN TERMS OF WICK POLYNOMIALS

We discuss two specific properties of Wick polynomial moving averages formed with the smeared Q-noise $w_\varepsilon(t)$ of Lemma 1. The symbol: $w_\varepsilon(t_1) \cdots w_\varepsilon(t_n)$: will be used for the Wick polynomial of operators $w_\varepsilon(t_1),..., w_\varepsilon(t_n)$ that have the ordered Gaussian property in their expectations. It may be defined compactly as follows:

$$:w_\varepsilon(t_1) \cdots w_\varepsilon(t_n): = \Bigg\{ \text{the coefficient of } a_1 a_2 \cdots a_n \text{ in } \exp\left[\sum_{k=1}^{n} a_k w_\varepsilon(t_k)\right] \Big/ \left\langle \exp\left[\sum_{k=1}^{n} a_k w_\varepsilon(t_k)\right] \right\rangle \Bigg\} \tag{5.1}$$

Wick polynomials exist as symmetric operators (see Appendix B). Wick polynomials of different degrees are orthogonal with respect to the inner product $(X, Y) \equiv \langle X^* Y \rangle$. Moreover, the totality of them spans the same linear space as does the ordinary polynomials. Some relevant details of the properties of Wick polynomials are given in Appendix B.

Theorem 2. Let $\eta(\tau_1,\dots,\tau_n)$ and $\zeta(\tau_1,\dots,\tau_n)$ for $n \geqslant 1$ be real and permutation-invariant, and belong to the class $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions with Fourier transforms,

$$\begin{aligned}\begin{pmatrix} H \\ Z \end{pmatrix} &(\omega_1,\dots,\omega_n) \\ &\equiv \mathscr{F}\left[\begin{pmatrix} \eta \\ \zeta \end{pmatrix}(\tau_1,\dots,\tau_n)\right] \\ &\equiv (2\pi)^{-n/2}\int_{-\infty}^{\infty} d\tau_1 \cdots \int_{-\infty}^{\infty} d\tau_n \begin{pmatrix} \eta \\ \zeta \end{pmatrix}(\tau_1,\dots,\tau_n) \\ &\quad\times \exp[-i(\omega_1\tau_1 + \cdots + \omega_n\tau_n)] \end{aligned} \tag{5.2}$$

that are polynomially bounded functions.[12] Let $\varepsilon(\tau) \in \mathscr{S}(\mathbb{R})$ [$\mathscr{S}(\mathbb{R})$ is the set of functions of rapid decrease]. Define, moreover, the moving average operators

$$\begin{aligned} A(s) &= \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{\infty} ds_n\, \eta(s-s_1,\dots,s-s_n) :w_\varepsilon(s_1)\cdots w_\varepsilon(s_n): \\ B(t) &= \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n\, \zeta(t-t_1,\dots,t-t_n) :w_\varepsilon(t_1)\cdots w_\varepsilon(t_n): \end{aligned} \tag{5.3}$$

Then the covariance function $C_\varepsilon(s-t) = \langle A^{(*)}(s)\, B(t)\rangle$ converges as a tempered distribution in $s-t$, when $\varepsilon(\tau)$ converges to $\delta(\tau)$ in the following sense: $E(\omega) = \mathscr{F}[\varepsilon(\tau)]$ converges to $(2\pi)^{-1/2}$ pointwise and $\exists c > 0$ such that $|E(\omega)| < c$, $\forall \omega \in \mathbb{R}$.[13]

The proof is given in Appendix C together with the limiting form of $C_\varepsilon(s-t)$. Since the differentiation of $A(s)$ of (4.13) involves the transformation

$$\mathscr{F}(\eta) \to \mathscr{F}[\partial\eta(s-s_1,\dots,s-s_n)/\partial s] \propto (\omega_1 + \cdots + \omega_n)\, H(\omega_1,\dots,\omega_n)$$

we have the following result:

Corollary 3. The statement in Theorem 2 also holds when $A(s)$ and $B(t)$ are replaced by $d^pA(s)/ds^p$ and $d^qB(t)/dt^q$, respectively $(p, q \geqslant 0)$.

[12] $\eta(\tau_1,\dots,\tau_n) \in \mathscr{S}'(\mathbb{R}^n)$ holds if and only if η or $H(\omega_1,\dots,\omega_n) \equiv \mathscr{F}(\eta)$ is a distributional derivative of a polynomially bounded, continuous function. Thus, η and ζ here must belong to a subclass of $\mathscr{S}'(\mathbb{R}^n)$, whose partial characterization is given in Corollary 3.

[13] This mode of convergence $E(\omega) \to (2\pi)^{-1/2}$ will be called the bounded convergence.

The result of the rearrangement of a polynomial of $w_\varepsilon(t)$'s into Wick polynomials of (5.1) depends explicitly on $\varepsilon(\tau)$. This is seen by an example,

$$w_\varepsilon(s)\, w_\varepsilon(t) = {:}w_\varepsilon(s)\, w_\varepsilon(t){:} + \langle w_\varepsilon(s)\, w_\varepsilon(t)\rangle$$

Since a usual polynomial is more easily constructed [say, in solving (2.5) in the form of perturbation series for polynomial $V(Q)$, putting aside the problem of convergence], the following criterion will be useful:

Lemma 4. Let $\xi(\tau_1,..., \tau_n)$ for $n \geqslant 2$ be real and permutation-invariant with a continuous Fourier transform $\Xi(\omega_1,..., \omega_n)$ satisfying the estimate

$$|\Xi(\omega_1,..., \omega_n)| \leqslant \text{const} \times \prod_{k=1}^{n} (1 + |\omega_k|)^{-1-g}, \qquad \exists g > 0 \tag{5.4}$$

The moving average operator with $\varepsilon(\tau) \in \mathscr{S}(\mathbb{R})$,

$$X(t) = \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n\, \xi(t - t_1,..., t - t_n)\, w_\varepsilon(t_1) \cdots w_\varepsilon(t_n) \tag{5.5}$$

can be decomposed into a finite sum of Wick polynomials that respectively give convergent contributions in forming a covariance function, say $\langle A^*(s)\, X(t)\rangle$ with $A(s)$ of Theorem 2, in the same limit of bounded convergence $E(\omega) \to (2\pi)^{-1/2}$.

The proof is again deferred to Appendix C. In the above statement, (5.4) and the continuity of $\Xi(\omega_1,..., \omega_n)$ form a sufficient condition that $X(t)$ of (5.5) converges (in the sense of the strong graph limit[9]) to a sum of well-defined operators (called multiple Wiener integrals[12]; see Appendix B) as $\varepsilon(\tau) \to \delta(\tau)$. The continuity of Ξ is in turn realized by, e.g., $\xi(\tau_1,..., \tau_n) \in L^1(\mathbb{R}^n)$. The restriction may be tight, but Corollary 3 gives another.

Corollary 5. For $X(t)$ of (5.5) in Lemma 4 and for any integer $r > 0$, the operator $d^r S(t)/dt^r$ exists and gives convergent covariance tempered distributions, just as $A(s)$ and $B(t)$ in Theorem 2 for the bounded convergence $E(\omega) \to (2\pi)^{-1/2}$.

It might be recalled that the assumption $\varepsilon(\tau) \in \mathscr{S}(\mathbb{R})$ assures the existence of $d^r X(t)/dt^r$ as an operator, as seen by partial integration in (5.5). However, it may lose such a sense in the limit $\varepsilon(\tau) \to \delta(\tau)$.

6. COMMENTS

Starting from Hamiltonian systems, we have shown the existence of a class of SIDEs and corresponding Q-SIDEs as reduced evolution equations. Their outstanding features, stated in results 1–3 of Section 1, may now be summarized as follows. (I) In both mechanics their forms are the same, including the fluctuation-dissipation relation. (II) Quantum noise forces are in the Lewis–Thomas class[10] with the KMS property with respect to the time shift. (III) Forms and properties of noise forces and dissipation terms [which form (quantum) heat reservoirs in pairs] do not depend on systems to which they are attached. (IV) For infinite systems with absolutely continuous $\Gamma(\omega)$'s, these noise forces admit (by Lemma 1) moving average representations in terms of the (quantized) Gaussian white noise, the representation kernel being identical in both mechanics. We may further point out by (3.10) and (4.9) that the continuum limit, for problems with polynomial $V(Q) \geqslant 0$ and for absolutely continuous $\Gamma(\omega)$, is realized by taking the smeared noise $w_\varepsilon(t)$ in the form of Lemma 1 and performing the limit $\varepsilon(\tau) \to \delta(\tau)$ on

$$dQ/dt = P/m$$

$$dP/dt = -V'(Q) - 2\gamma \int_{-\infty}^{t} C(t-s)\, P(s)\, ds + \sigma w_\varepsilon(t) \tag{6.1}$$

$$C(\tau) = \int_{-\infty}^{\infty} \varepsilon(s+\tau)\, \varepsilon(s)\, ds$$

Since a Brownian particle can have only a finite, even if very large, mass ratio $m/\tilde{m}$ with respect to surrounding particles, the physical reality corresponds to Q-SIDE (2.5) or (6.1), where $\{Q(t), P(t)\}$ are well-defined observables, an observable A being defined here as a self-adjoint operator with the state vector in its domain (i.e., $\langle A^2 \rangle < \infty$), in analogy to square-integrable random variables. Our interest is in the covariance functions of observables, but the setting of Brownian motion furhter selects relevant ones that converge in the continuum limit. Not all covariance functions associated with observables of (2.5) or (6.1) can have this property. Conversely, the convergence may arise with observables that lose their sense in the limit. As an example, take the classic Q-OU problem for (3.1). The solution of (2.5) or (6.1) is then obtained with Laplace transforms (Lemma 4 of Ref. 17) as moving averages of Q-noise. It confirms with (4.5) and (4.6) the conclusions[3] that $Q(t)$ remains an observable in the continuum limit, but the moving average kernel of $P(t)$ leaves the class $\mathscr{T}$ of (4.7). Thus, $\langle P^2(t) \rangle$ (hence $\langle |P^n| \rangle$ for $n > 2$) diverges in the limit, implying nonconvergence of e^{iaP} for $a \in \mathbb{R}$ (cf. the proof of Proposition 6 in

Ref. 17). We must admit that P and $E = P^2/2m$ lose their physical relevance despite the classical $\langle E \rangle_c \equiv 1/2\beta$; compare also the arguments in Ref. 3 to renormalize E with Section 6 of Ref. 12, which seem inconclusive. We now take Theorem 2, with $B(t) = :P^n(t):$ for Q-OU $P(t)$ of (6.1). This $B(t)$ loses the sense of an observable as $\varepsilon(\tau) \to \delta(\tau)$ by (B.5). But a vast class of $A(s)$ gives

$$Y(t) = \left[\int_{-\infty}^{\infty} d\tau\, \phi(\tau)\, A(t+\tau) \right] B(t) \qquad \text{for} \quad \phi(\tau) \in \mathscr{S}$$

with $\langle Y(t) \rangle$ convergent in this limit. The limit of $\langle Y(t) \rangle$ is a quantity of physical relevance that approximates the one for (6.1), and should be evaluated by Q-SDE (2.10). However, this relevance seems unsubstantiated with the framework of (2.10) alone. These indicate the following circumstance: Q-SIDE (2.5) or (6.1) determines what remains significant in the continuum limit, Q-SDE (2.10) should in general refer to (2.5) in this regard, but possibilities remain to be pursued for Q-SDE (2.10) to have a formulation that directly comprises convergent covariance functions of such nonobservables as $Y(t)$.

We have stated at the end of Section 2 that the classical SDE (2.10) is an idealization of the physically more realistic (2.5). It is well known that SDEs show great utility in constructing diffusion processes, due mainly to the powerful existence of stochastic integrals. This circumstance suggests a similar utility of Q-SDE (2.10) as an idealization of Q-SIDE (2.5) or (6.1), but the possibility of stochastic calculi on Q-noise still has a dim outlook. It is known[12] that Wick polynomials of Q-noise become commutative in the limit $\hbar \to 0$, and reproduce multiple Wiener integrals (iterated Itô stochastic integrals) based on the c-number Gaussian white noise $w(t)$. Lemma 1 of Section 4 shows the same basic role of the Q-noise among operator Gaussian processes as that of the c-number $w(t)$ among the usual Gaussian processes. However, these still remain as isolated analogies. Some breakthrough is awaited.

APPENDIX A. Q-NOISE

A cyclic representation of Q-noise, which is always meant in the present work, is given by a Hilbert space $\mathscr{H}$, a vector $\Phi \in \mathscr{H}$, and a set of unitary operators $\{e^{iw(\xi)}; \xi(t) \in \mathscr{T}\}$ on $\mathscr{H}$ that give the Weyl relation

$$e^{iw(\xi)} e^{iw(\eta)} = e^{iw(\xi+\eta)} e^{-[w(\xi), w(\eta)]/2}$$

as well as

$$\langle e^{iw(\xi)} \rangle = (\Phi, e^{iw(\xi)} \Phi) = e^{-s(\xi,\xi)/2}$$

with the inner product (X, Y) of $\mathscr{H}$ and with a linear span of $\{e^{iw(\xi)}\Phi; \xi \in \mathscr{T}\}$ dense in $\mathscr{H}$. Here the test function space $\mathscr{T}$ is given by (4.7) as the broadest class that gives sense to both $\sigma(\xi, \eta)$ and $s(\xi, \eta)$ of (4.5) and (4.6). Such a representation exists by the general theory,[11] which is applicable here by $\mu(\omega) \geqslant \beta\hbar\, |\omega|/2$, which gives

$$|\sigma(\xi, \eta)| \leqslant [s(\xi, \xi)\, s(\eta, \eta)]^{1/2}, \qquad \forall \xi, \eta \in \mathscr{T} \tag{A.1}$$

It is well known and easy to see that (4.5) and (4.6) imply

$$\begin{aligned} &\langle w(\xi_1) \cdots w(\xi_{2n+1}) \rangle = 0 \\ &\qquad \langle w^{(*)}(\xi)\, w(\eta) \rangle = s(\xi, \eta) + i\sigma(\xi, \eta) \equiv \langle\!\langle \xi, \eta \rangle\!\rangle \end{aligned} \tag{A.2}$$

together with the quasi-free (ordered Gaussian) law of decomposition of $\langle w(\xi_1) \cdots w(\xi_{2n}) \rangle$ to ordered pair expectations. We now state the following result[6,7]:

Proposition A. Let the time shift τ_t on the polynomials of $\{w(\xi); \xi \in \mathscr{T}\}$ be defined by $\tau_t w(s) = w(s+t)$ or

$$\begin{aligned} &\tau_t[w(\xi_1) \cdots w(\xi_n)] \\ &\qquad = w\{\mathscr{F}^{-1}[e^{-i\omega t}\Xi_1(\omega)]\} \cdots w\{\mathscr{F}^{-1}[e^{-i\omega t}\Xi_n(\omega)]\} \end{aligned} \tag{A.3}$$

Any polynomial of Q-noise has the β-KMS property with respect to τ_t.

The proof is sketched to make the paper self-contained. By definition, τ_t respects the algebraic structure of a polyomial of Q-noise operators. The quasi-free law with (A.2) implies that Φ is τ_t-invariant, $(\Phi, \tau_t X\Phi) = (\Phi, X\Phi)$, for any polynomial X of Q-noise. Denote $X(0) = w(\xi)$ and $Y(t) = \tau_t w(\eta)$ for $\xi, \eta \in \mathscr{T}$. The following hold:

$$\begin{aligned} F(t) &= (\Omega, X(0)\; Y(t)\Omega) \\ &= \int_{-\infty}^{\infty} \Xi^*(\omega)\, H(\omega)[\beta\hbar\, |\omega|/(1 - e^{-\beta\hbar\,|\omega|})] \\ &\qquad \times e^{-i\omega[t - i\beta\hbar(1 + \operatorname{sign}\omega)]}\, d\omega \end{aligned} \tag{A.4}$$

$$F(t + i\beta\hbar) = (\Omega,\; Y(t)\, X(0)\Omega)$$

By the uniform convergence of the integral in (A.4) with respect to t, $F(z)$ is continuous for $0 \leqslant \operatorname{Im} z \leqslant \beta\hbar$ and analytic in the strip $0 < \operatorname{Im} z < \beta\hbar$. This proves the β-KMS property of any polynomial of $\{w(\xi); \xi \in \mathscr{T}\}$ with respect to τ_t, by the quasi-free law of decomposition of expectation values to those of pairs.

APPENDIX B. WICK POLYNOMIALS AND MULTIPLE WIENER INTEGRALS

The Wick polynomial[13-15] (5.1) is defined more explicitly below.

Definition B.1. Let $\xi_1,..., \xi_m \in \mathscr{T}$ be arbitrary, and define

$$\mathbf{G}_0(c) \equiv c, \qquad \forall c \in \mathbb{R}$$

$$\begin{aligned} \mathbf{G}_m(\xi_1 \xi_2 \cdots \xi_m) &\equiv :w(\xi_1)\, w(\xi_2) \cdots w(\xi_m): \\ &\equiv w(\xi_1)\, w(\xi_2) \cdots w(\xi_m) + \sum_{p=1}^{[m/2]} (-1)^p \\ &\quad \times \sum_{\substack{\text{all ways of taking } p \text{ pairs} \\ i(1)<j(1),..., i(p)<j(p) \\ \text{from } \{1,..., m\} \text{ with the remainder} \\ k(1)<k(2)<\cdots<k(m-2p)}} \\ &\quad \times \left(\prod_{q=1}^{p} \langle\!\langle \xi_{i(q)}, \xi_{j(q)} \rangle\!\rangle \right) w(\xi_{k(1)}) \cdots w(\xi_{k(m-2p)}), \qquad m \geqslant 1 \end{aligned} \tag{B.1}$$

For $\langle\!\langle \cdots \rangle\!\rangle$, see (A.2). The following properties are well known.

Proposition B.2. (A) $\mathbf{G}_m(\xi_1 \cdots \xi_m)$ is a symmetric operator on the domain $\mathscr{A}\Omega$ with

$$\mathbf{G}_m(\xi_{\pi(1)} \cdots \xi_{\pi(m)}) = \mathbf{G}_m(\xi_1 \cdots \xi_m) \tag{B.2}$$

where $\mathscr{A}$ is the totality of polynomials of $w(\xi)$ ($\xi \in \mathscr{T}$) and $\{\pi(1),..., \pi(m)\}$ is any permutation of $\{1,..., m\}$.

(B) Orthogonality holds with $\forall \eta_1,..., \eta_n \in \mathscr{T}$,

$$\langle \mathbf{G}_m^{(*)}(\xi_1 \cdots \xi_m)\, \mathbf{G}_n(\eta_1 \cdots \eta_n) \rangle = \delta_{mn} \sum_{\{\pi(1),...,\pi(m)\}} \prod_{k=1}^{m} \langle\!\langle \xi_k, \eta_{\pi(k)} \rangle\!\rangle$$

Let $\{\mathscr{T}, s\}$ denote the real Hilbert space $\mathscr{T}$ equipped with the inner product $s(\xi, \eta)$ of (4.6). Let $\bar{\mathscr{T}}^{\otimes m}$ denote the symmetric, m-fold tensor product Hilbert space[14] constructed with $\{\mathscr{T}, s\}$. Let $\{\Phi_k(\omega); k = 1, 2,...\}$

[14] Since $s(\xi, \eta)$ is diagonal in the frequency space, an element of $\mathscr{F}(\bar{\mathscr{T}}^{\otimes m})$ is a usual symmetric function $\Xi(\omega_1,..., \omega_m) = \Xi^*(-\omega_1,..., -\omega_m) \in L^2[\mathbb{R}^m, \mu(\omega_1) \cdots \mu(\omega_m)\, d\omega_1 \cdots d\omega_m]$.

be an orthonormal basis of $\mathscr{F}(\mathscr{T})$. Any $\xi \in \bar{\mathscr{T}}^{\otimes m}$ may be expressed as follows by its Fourier transform:

$$\Xi(\omega_1,..., \omega_m) = \lim_{K \to \infty} \sum_{k(1)=1}^{K} \cdots \sum_{k(m)=1}^{K} c[k(1) \cdots k(m)] \times \prod_{l=1}^{m} \Phi_{k(l)}(\omega_l) \tag{B.3}$$

where $c[k(1) \cdots k(m)]$ is a real coefficient. Define

$$G_m(\xi^{(K)}) \equiv \sum_{k(1)=1}^{K} \cdots \sum_{k(m)=1}^{K} c[k(1) \cdots k(m)] \, \mathbf{G}_m(\phi_{k(1)} \cdots \phi_{k(m)}) \tag{B.4}$$

We quote the following result,[12] which extends the algebraically defined Wick polynomials to a continuum $\bar{\mathscr{T}}^{\otimes m}$ of kernels $\xi(t_1,..., t_m)$.

Proposition B.3. For any $\xi \in \bar{\mathscr{T}}^{\otimes m}$, the strong graph limit[9] $K \to \infty$ of $G_m(\xi^{(K)})$ of (B.4) exists, especially on respective vectors of $\mathscr{A}\Omega$, and defines a symmetric, closed operator on $\mathscr{H}$ to be denoted $G_m(\xi)$, the orthogonal polynomial or the multiple Wiener integral of mth degree based on the Q-noise $w(t)$. There holds

$$\langle G_m^{(*)}(\xi) \, G_n(\eta) \rangle = \delta_{mn} m! \, \langle\!\langle \xi, \eta \rangle\!\rangle_m, \qquad \forall \eta \in \bar{\mathscr{T}}^{\otimes n}$$

$$\langle\!\langle \xi, \eta \rangle\!\rangle_m \equiv \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_m \Xi^*(\omega_1,..., \omega_m) \, H(\omega_1,..., \omega_m) \times \prod_{k=1}^{m} [\mu(\omega_k) - \tfrac{1}{2}\beta\hbar\omega_k] \tag{B.5}$$

$$\Xi(\omega_1,..., \omega_m) \equiv \mathscr{F} \xi(t_1,..., t_m)$$

APPENDIX C. PROOFS OF THEOREM 2 AND LEMMA 4

Proof of Theorem 2. Let $\mathscr{S}$ be Schwartz's space of rapidly decreasing functions and $\rho(s) \in \mathscr{S}$ be arbitrary. By (5.3) we have

$$\int_{-\infty}^{\infty} \rho(s) \, A(s) \, ds = (2\pi)^{(n+1)/2} \, G_n[R(\omega_1 + \cdots + \omega_n) \, H(\omega_1,..., \omega_n) \, E(\omega_1) \cdots E(\omega_n)]$$

$$B(t) = (2\pi)^{n/2} \, G_n\{\exp[-i(\omega_1 + \cdots + \omega_n)t] \times Z(\omega_1,..., \omega_n) \, E(\omega_1) \cdots E(\omega_n)\}$$

$$R(\omega) = \mathscr{F}[\rho(s)]$$

These and (B.5) give

$$\int_{-\infty}^{\infty} ds\, \rho(s) \langle A^{(*)}(s)\, B(t) \rangle$$

$$= n!\, (2\pi)^{n+1/2} \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_n$$

$$\times \{ R(\omega_1 + \cdots + \omega_n) \exp[i(\omega_1 + \cdots + \omega_n)t] \}^*$$

$$\times H^*(\omega_1, \ldots, \omega_n)\, Z(\omega_1, \ldots, \omega_n) \prod_{k=1}^{n} |E(\omega_k)|^2 [\mu(\omega_k) - \tfrac{1}{2}\beta\hbar\omega_k] \quad \text{(C.1)}$$

This proves the form $\langle A^{(*)}(s)\, B(t) \rangle = C_\varepsilon(s-t)$. By the estimate

$$0 < \mu(\omega) - \frac{1}{2}\beta\hbar\omega \leqslant \begin{cases} \text{const} \times (1 + |\omega|)\, e^{-\beta\hbar|\omega|}, & \omega \geqslant 0 \\ \text{const} \times (1 + |\omega|), & \omega < 0 \end{cases}$$

we need only to show that the following integral exists, converges as $E(\omega) \to (2\pi)^{-1/2}$ boundedly, and converges to zero as $R(\omega) \to 0$ in $\mathscr{S}$:

$$I_k(\rho) \equiv \int_{-\infty}^{0} d\omega_1 \cdots \int_{-\infty}^{0} d\omega_k \int_{0}^{\infty} d\omega'_1 \cdots \int_{0}^{\infty} d\omega'_{n-k} \prod_{l=1}^{k} (1 + |\omega_l|)$$

$$\times |R(-\omega_1 - \cdots - \omega_k + \omega'_1 + \cdots + \omega'_{n-k})|$$

$$\times |H(\omega_1, \ldots, \omega_k, \omega'_1, \ldots, \omega'_{n-k})$$

$$\times E(\omega_1) \cdots E(\omega'_{n-k})|^2 \prod_{l=1}^{n-k} (1 + |\omega'_l|)$$

$$\times \exp[-\beta\hbar(\omega'_1 + \cdots + \omega'_{n-k})]$$

$k = 0, 1, \ldots, n$. Denote $\omega = -(\omega_1 + \cdots + \omega_k)$, $\omega' = \omega'_1 + \cdots + \omega'_{n-k}$. Since $|H|$ is polynomially bounded, we have

$$|I_k(\rho)| \leqslant \text{const} \times \int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega'$$

$$\times \left(\int \cdots \int_{-(\omega_1 + \cdots + \omega_k) \leqslant \omega} \right) \left(\int \cdots \int_{\omega'_1 + \cdots + \omega'_{n-k} \leqslant \omega'} \right)$$

$$\times (1 + \omega)^a (1 + \omega')^a\, |R(\omega' - \omega)|\, e^{-\beta\hbar\omega'}$$

$$\leqslant \text{const} \times \left[\int_{0}^{\infty} d(\omega' - \omega) \int_{0}^{\infty} d\omega\, e^{-\beta\hbar\omega} \right.$$

$$\left. + \int_{0}^{\infty} d(\omega - \omega') \int_{0}^{\infty} d\omega'\, e^{-\beta\hbar\omega'} \right]$$

$$\times (1 + \omega + |\omega - \omega'|)^b\, |R(\omega' - \omega)| \quad \text{(C.2)}$$

The rhs converges by $R(\omega) \in \mathscr{S}$. This proves the integrability. The convergence for $E(\omega) \to (2\pi)^{-1/2}$ follows by the dominated convergence theorem. That $I_k(\rho) \to 0$ for $\rho \to 0$ in $\mathscr{S}$ is manifest by (C.2). ▮

Proof of Lemma 4. Let $\xi(\tau_1, \ldots, \tau_n)$ have the property stated in Lemma 4; (5.4) implies $\xi \in \bar{\mathscr{T}}^{\otimes m}$. Definition (B.1) at once gives the following reduction formula[12]:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n\, \xi(t-t_1, \ldots, t-t_n)\, w_\varepsilon(t_1) \cdots w_\varepsilon(t_n) \\
&\quad = \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n\, \xi(t-t_1, \ldots, t-t_n) :w_\varepsilon(t_1) \cdots w_\varepsilon(t_n): \\
&\qquad - \sum_{k=1}^{[n/2]} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_{n-2k}\, \xi_k(t-t_1, \ldots, t-t_{n-2k}) \\
&\qquad \times w_\varepsilon(t_1) \cdots w_\varepsilon(t_{n-2k})
\end{aligned}$$

$$\begin{aligned}
&\xi_k(\tau_1, \ldots, \tau_{n-2k}) \\
&\quad \equiv (-1)^k [n!!/(n-2k)!!] \int_{-\infty}^{\infty} d\tau'_1 \cdots \int_{-\infty}^{\infty} d\tau'_{2k} \\
&\qquad \times \xi(\tau_1, \ldots, \tau_{n-2k}, \tau'_1, \ldots, \tau'_{2k}) \prod_{l=1}^{k} \langle w_\varepsilon(\tau'_{2l-1})\, w_\varepsilon(\tau'_{2l}) \rangle \\
&\quad = (2\pi)^{-(n-2k)/2} \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_{n-2k}\, \Xi_k(\omega_1, \ldots, \omega_{n-2k}) \\
&\qquad \times \exp(i\omega_1\tau_1 + \cdots + i\omega_{n-2k}\tau_{n-2k})
\end{aligned}$$

$$\begin{aligned}
&\Xi_k(\omega_1, \ldots, \omega_{n-2k}) \\
&\quad = (-1)^k [n!!/(n-2k)!!] \int_{-\infty}^{\infty} d\omega'_1 \cdots \int_{-\infty}^{\infty} d\omega'_k \\
&\qquad \times \Xi(\omega_1, \ldots, \omega_{n-2k}, \omega'_1, -\omega'_1, \omega'_2, -\omega'_2, \ldots, \omega'_k, -\omega'_k) \\
&\qquad \times \prod_{l=1}^{k} [\mu(\omega'_l) - \tfrac{1}{2}\beta\hbar\omega'_l]\, |E(\omega'_l)|^2 \qquad \text{(C.3)}
\end{aligned}$$

For each k, $\Xi_k(\omega_1, \ldots, \omega_{n-2k})$ exists by (5.4), and converges in $\mathscr{F}(\bar{\mathscr{T}}^{\otimes(n-2k)})$ as $E(\omega'_l) \to (2\pi)^{-1/2}$ boundedly. Proposition B.3 proves the assertion of Lemma 4. ▮

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